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A BINARY SHEFFER OPERATOR WHICH DOES THE WORK OF QUANTIFIERS AND SENTENTIAL CONNECTIVES

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In recent years, the range of propositional systems for which binary Sheffer operators have been discovered has broadened to include various systems with multiple truth-values, modalities, and multigrade connectives (see [1] for a review). In this paper*, I present an indigenously definable binary Sheffer operator for the first order predicate calculus, and show how the technique employed there to combine quantifiers and sentential connectives in a single operator can be used to extend the previously discovered binary Sheffer operators to capture *quantified* modal systems.

We consider a stroke language containing a countable number of individual variables $x_1, x_2, \ldots$, and for each positive integer n, a countable number of n-ary predicates. If P is an n-ary predicate letter, the result of concatenating P to the left of n variable letters is a well-formed formula. If A and B are wffs, A/B is a wff. For any positive integer k and any wff A, we introduce the notation A^k (the name of a formula) as follows:

$$A^1 = A/(A/A)$$
$$\text{if } A^n = B/C, \text{ then } A^{n+1} = C/(C/C).$$

We interpret the stroke language in terms of a standard predicate language with the usual sentential connectives and quantifiers as follows. For any wffs A and B, let

$$A/B \rightleftharpoons (v) \sim (A \cdot B)$$

where $\rightleftharpoons$ indicates semantic equivalence, and v is an individual variable such that:

(i) $v = x_k$ iff $B = A^k$
(ii) v is alphabetically the first variable which does not occur in A or B iff for all k, $B \neq A^k$.

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On the basis of this equivalence, we may redefine the usual connectives and quantifiers in the stroke language.

$$\sim A =_{Df} A/A$$
$$(\exists x_k)A =_{Df} \sim (A/A^k)$$
$$A \cdot B =_{Df} \begin{cases} (A/B)/(A/B) & \text{iff for all } k,\ B \neq A^k \\ A & \text{iff for some } k,\ B = A^k \end{cases}$$

The correctness of these definitions follows directly from the interpretation of the stroke in terms of the universal quantifier, negation, and conjunction. The definition of the existential quantifier is justified by the fact that no matter what wff A is, the formula A^k is valid. We must separate cases in the definition of $A \cdot B$, since if $B = A^k$ and x_k occurs free in A, our interpretation of the stroke requires binding x_k in A/A^k, changing the sense so that $A \cdot A^k \not\rightleftharpoons (A/A^k)/(A/A^k)$. Since A^k is valid no matter what wff A is, $A \cdot A^k \rightleftharpoons A$, as in the second clause of our definition of conjunction. Clearly the binary stroke operator under this interpretation is adequate for first order quantification theory.

This stroke operator thus extends Sheffer's original binary connective for standard propositional languages to quantified languages by using a general technique. First, Sheffer's non-conjunction connective is prefaced by a universal quantifier. If this quantification is vacuous for a given pair of wffs and a particular variable, the original connective results, and negation and conjunction can be defined as before. So some way is needed to keep track of the variable of quantification. Since the quantifier we want to define need apply to only one formula at a time, and since we have a two-place operator to work with, we can syntactically code the variable of quantification into a formula which occupies the second place, in such a way as not to collide with the treatment of conjunction (the only two-place connective we have to represent). This same technique can be used to extend other propositional Sheffer connectives to treat quantifiers. For instance, we can generalize Massey's binary connective for propositional S5 to a binary operator for quantified S5. We consider a language syntactically isomorphic to the stroke language, save that A/B is replaced everywhere by $A*B$. We interpret the asterisk language in terms of the usual connectives, quantifiers, and modal operators as follows:

$$A*B \rightleftharpoons (v)[\sim\Diamond(A \cdot B) \vee \Diamond(A \cdot B) \cdot \Diamond(A \cdot \sim B) \cdot \sim(A \cdot \sim B) \vee \Diamond(A \cdot B) \cdot \sim\Diamond(A \cdot \sim B) \cdot \sim(A \cdot B)]$$

where A and B are any wffs and v is an individual variable defined by conditions (i) and (ii) above (using an account of A^k in which the asterisk replaces the stroke, of course). Then in the asterisk language we can define the connectives, quantifiers, and modal operators:

$$\sim A =_{Df} A * A$$
$$(\exists x_k)A =_{Df} \sim(A*A^k)$$
$$A \supset B =_{Df} \begin{cases} A*(A*B) & \text{iff for all } k,\ B \neq A^k \\ A*(A*A) & \text{iff for some } k,\ B = A^k \end{cases}$$
$$\Diamond A =_{Df} ((\sim A * A) * A) \supset A$$

The proofs are as in [1], save for the existential quantifier and material implication. Since for any wff A, A^k is always valid on this interpretation of the asterisk, $A \supset A^k$ will always be equivalent to any A^i, in particular to A^1, as in the second clause of the definition of the horseshoe. $A{*}A^k$ is obviously equivalent to $(x_k){\sim}A$ on our interpretation, so the definition of the existential quantifier is correct as well.

Massey's binary Sheffer connective for propositional S4 (reported in [2]), and Sobociński's simplified version of the same connective[1] (reported in [3]), can be extended to binary operators for quantified S4 by the same technique.

REFERENCES

[1] Hendry, H. E. and G. J. Massey, "On the concepts of Sheffer functions" in K. Lambert, *The Logical Way of Doing Things*, Yale University Press, New Haven (1969), pp. 279-293.

[2] Massey, G. J., "Binary closure-algebraic operations that are functionally complete," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 340-342.

[3] Sobociński, B., "Note on G. J. Massey's closure-algebraic operation," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 343-346.

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1. The definition of the simplified S4 connective which actually appears in [3] is incorrect, due to a faulty transliteration into the language of modal logic of the closure-algebraic results presented there. The correct definition of the simplified S4 connective (*W*1 on page 346) is *EWpqKCNpCMpMqCpNq*.